430

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CONTENTS

	Pages.
K. Nagabhushanam: On a property of the Lagrangian	89—90
S. C. Dhar: On certain Self-Reciprocal Functions ..	91—96
V. Ganapathy Iyer: On Maximal Integral Functions	97—115
S. B. Krishna Murti: A Set of Axioms for Topological Algebra..	116—119
Haridas Bagchi: A Note on Cyclides ..	120—124
K. Sambasiva Rao: A correction	125

1457

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ON A PROPERTY OF THE LAGRANGIAN

By K. NAGABHUSHANAM, *Andhra University.*

[Received 2 October 1939]

In affine manifold of states and time A_{2n+1}, we have the following facts.

1. The trajectories of a dynamical system with n degrees of freedom are given by the singular curves of the curl of (X_i), i.e. by

$$a_{ik}\, dx^k = 0,^* \quad (i, k = 1, 2, \ldots, 2n+1), \qquad (a)$$

where $a_{ik} = \dfrac{\partial X_i}{\partial x^k} - \dfrac{\partial X_k}{\partial x^i}$ and the summation convention is used for a repeated index, provided the rank of the matrix $|\, a_{ik}\,;\, X_k|$ is $2n+1$.

2. If the cofactors of a_{ik} in the determinant $|a_{ik}|$ are denoted by A^{ik}, then $A^{ik} = \alpha^i\, \alpha^k$†.

3. (α^i) is a contravariant vector density co-directional with the trajectories.‡

4. By using coordinates in which Lagrange's brackets vanish, i. e.

$$[x^j x^{2n+1}] = 0, \; (j = 1, 2, .., 2n)$$

we obtain

$$\operatorname{div} \alpha^i = \alpha^i{}_{,i} = 0, \S \qquad (b)$$

the comma denoting covariant differentiation with respect to the symmetric connection.

5. $X_i \alpha^i$ is a Jacobi's multiplier‖ for the equations of the trajectories which may be written as

$$\frac{dx^k}{\alpha^k / X_i \alpha^i} = \varepsilon, \; (k = 1, 2, \ldots, 2n+1). \qquad (c)$$

*K. Nagabhushanam: Transformation theory of dynamics in the manifold of states and time, *Jour. Ind. Math. Soc.* (1) 20 (1933), 237.

† G. Kowalewski: *Determinanten-theorie,* p. 124.

‡ K. Nagabhushanam: On the form $\sum_{r=1}^{n} p_r\, dq^r - Hdt$. *Proc. Ind. Ac. Sc.* 1, 557.

§ K. Nagabhushanam: Action form and Jacobi's multiplier (to be published).

‖ K. Nagabhushanam: A property of Jacobi's multiplier, *Jour. Lond. Math. Soc.* 9, 15.

6. $\iint\ldots\int\left(X_i\alpha^i\right)d\omega$ is an integral-invariant attached to (c)*, where $d\omega = dx^1 dx^2 \ldots dx^{2n+1}$.

In what follows we shall obtain two special forms of this integral invariant.

If t is a suitable function† of (x^i), it can be regarded as a time measure, and the Lagrangian L can be invariantly defined‡ for this time measure as

$$L = X_i\xi^i \Big/ \frac{\partial t}{\partial x^k}\xi^k, \tag{d}$$

where (ξ^i) is a vector co directional with the trajectories. Since L is homogeneous of degree zero in the ξ's, we can substitute proportional quantities, (α^i). Thus

$$L = X_i\alpha^i \Big/ \frac{\partial t}{\partial x^k}\alpha^k. \tag{e}$$

From (e) we see that

$$\iint\ldots\int\left(L\frac{\partial t}{\partial x^i}\alpha^i\right)d\omega \tag{f}$$

is an integral-invariant. Let us consider variables in which $x^{2n+1} = t$. In all such systems, the quantity

$$\frac{\partial t}{\partial x^k}\alpha^k = \alpha^{2n+1}$$

is a scalar density, and (f) becomes $\iint\ldots\int\left(L\alpha^{2n+1}\right)d\omega$. $\quad (g)$

If the coordinates are further specialised such that

$$x^r = q^r,\ x^{r+n} = p_r,\ x^{2n+1} = t,$$

and $\quad X_i dx^i = p_r dq^r - H dt \quad \begin{pmatrix} i = 1, 2, \ldots, 2n+1 \\ r = 1, 2, \ldots, n \end{pmatrix}$

then $\alpha^{2n+1} = \pm 1$ and L is found to be $p_r\dfrac{\partial H}{\partial p_r} - H$.

Thus (g) now becomes

$$\pm\iint\ldots\int\left(p_r\frac{\partial H}{\partial p_r} - H\right)dq^1\ldots dq^n dp_1\ldots dp_n dt. \tag{h}$$

We have thus obtained the two forms (g) and (h) of the integral-invariant of order $2n+1$ connected with the Lagrangian in systems of variables with $x^{2n+1} = t$.

* E. Goursat: *Lecons sur le probleme de Pfaff*, p. 221.

† K. Nagabhushanam: On the form $\sum_{r=1}^{n} p_r dq^r - Hdt$. loc. cit., p. 556.

‡ *See* Ref. †, p. 560.

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ON CERTAIN SELF-RECIPROCAL FUNCTIONS*

By S. C. DHAR, *Nagpur University*.

[Received 31 October 1939]

1. *Introduction.* In a recent paper† I obtained certain self-reciprocal functions in the Hankel-transform by the help of Hardy-Titchmarsh formulæ. These transforms can also be studied by the help of functional equations, used by other workers.‡ In the present paper I investigate some other types of self-reciprocal functions.

Following Hardy and Titchmarsh§, we say that a function $\phi(x)$ is self-reciprocal in the Hankel-transform, if

$$\phi(x)=\int_0^\infty (xt)^{\frac{1}{2}}J_\nu(xt)\phi(t)\,dt, \tag{1}$$

when $\nu > \pm\frac{1}{2}$, and $\phi(x)$ will be called R_ν. If however, $\nu=\pm\frac{1}{2}$, (1) reduces to the Fourier sine or cosine transform. We investigate here Whittaker functions which are self-reciprocal in the Hankel-transform. By the help of the infinite integral of the W-function, we show in **5** that the function

$$x^{\nu-2k+\frac{1}{2}}e^{-\frac{1}{4}x^2}W_{k,\,k-\frac{1}{2}}(\tfrac{1}{2}x^2)$$

is R_ν. So also is the function

$$x^{\nu-2k+\frac{1}{2}}e^{-\frac{1}{4}x^2}W_{k,-k+\frac{1}{2}}(\tfrac{1}{2}x^2).$$

By the same method we further show in **6** that

$$x^{\nu\mp 2m-\frac{1}{2}}e^{\frac{1}{4}x^2}W_{-\nu\pm 3m-\frac{1}{2},\,m}(\tfrac{1}{2}x^2) \text{ are } R_\nu.\,\|$$

* Read before the Science Congress held at Lahore, 1939.

† Dhar: On certain Functions which are self-reciprocal in the Hankel-transform, *Jour. London Math. Soc.* 14 (1939), 30-2.

‡ R. S. Varma: Some Functions which are self-reciprocal in the Hankel-transform, *Proc London Math. Soc.* (2), 42 (1937), 9-17.

§ Hardy and Titchmarsh; *Proc. London Math. Soc.* (2), 33 (1931), 225-32.

‖ Dhar: *loc. cit.*

2. Let

$$f(s)=\int_0^\infty x^{s-1}e^{-\frac{1}{4}x^2}W_{k,\,m}(\tfrac{1}{2}x^2)\,dx. \tag{2.1}$$

Now we know* that

$$W_{k,m}(x)=\frac{e^{-\frac{1}{2}x}x^{m+\frac{1}{2}}}{\Gamma(\frac{1}{2}-k+m)}\int_0^\infty t^{-k+m-\frac{1}{2}}(1+t)^{k+m-\frac{1}{2}}e^{-xt}dt,$$
$$R(k-m-\tfrac{1}{2})\leqslant 0.$$

Hence we have

$$f(s)=\frac{1}{2^{m+\frac{1}{2}}\Gamma(\frac{1}{2}-k+m)}$$
$$\times\int_0^\infty x^{s+2m}dx\int_0^\infty t^{-k+m-\frac{1}{2}}(1+t)^{k+m-\frac{1}{2}}e^{-\frac{1}{2}x^2(t+1)}dt$$
$$=\frac{1}{\Gamma(\frac{1}{2}-k+m)2^{m+\frac{1}{2}}}$$
$$\times\int_0^\infty t^{-k+m-\frac{1}{2}}(1+t)^{k+m-\frac{1}{2}}dt\int_0^\infty x^{s+2m}e^{-\frac{1}{2}x^2(t+1)}dx,$$

obtained by changing the order of integration which is easily seen to be permissible. Now we put $z=\frac{1}{2}x^2$ and obtain

$$f(s)=\frac{2^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}-k+m)}\int_0^\infty t^{-k+m-\frac{1}{2}}(1+t)^{k+m-\frac{1}{2}}dt\int_0^\infty z^{m+\frac{1}{2}s-\frac{1}{2}}e^{-z(t+1)}dz$$
$$=\frac{2^{\frac{1}{2}s-1}}{\Gamma(\frac{1}{2}-k+m)}\int_0^\infty t^{-k+m-\frac{1}{2}}(1+t)^{k-\frac{1}{2}s-1}dt\int_0^\infty y^{m+\frac{1}{2}s-\frac{1}{2}}e^{-y}dy$$
(on transformation)
$$=\frac{2^{\frac{1}{2}s-1}\Gamma(m+\frac{1}{2}s+\frac{1}{2})}{\Gamma(\frac{1}{2}-k+m)}\int_0^\infty t^{-k+m-\frac{1}{2}}(1+t)^{k-\frac{1}{2}s-1}dt,$$
$$R(m+\tfrac{1}{2}s+\tfrac{1}{2})>0.$$
$$=2^{\frac{1}{2}s-1}\frac{\Gamma(m+\frac{1}{2}s+\frac{1}{2})\,\Gamma(\frac{1}{2}s-m+\frac{1}{2})}{\Gamma(\frac{1}{2}s-k+1)},$$

when $R(-k+m+\frac{1}{2})>0$, $R(k-\frac{1}{2}s-1)<0$ and $R(\frac{1}{2}s-m+\frac{1}{2})>0$. Therefore we have

$$f(s)=2^{\frac{1}{2}s-1}\frac{\Gamma(m+\frac{1}{2}s+\frac{1}{2})\,\Gamma(\frac{1}{2}s-m+\frac{1}{2})}{\Gamma(\frac{1}{2}s-k+1)}, \tag{2.2}$$

where $0<R(k-\frac{1}{2})<R(m)<R(\frac{1}{2}s+\frac{1}{2})$.

3. If now $x^\lambda e^{-\frac{1}{4}x^2}W_{k,\,m}(\frac{1}{2}x^2)$ is R_ν, so that

$$x^\lambda e^{-\frac{1}{4}x^2}W_{k,\,m}(\tfrac{1}{2}x^2)=\int_0^\infty (xy)^{\frac{1}{2}}J_\nu(xy)\,y^\lambda e^{-\frac{1}{4}y^2}W_{k,\,m}(\tfrac{1}{2}y^2)\,dy \tag{3.1}$$

* Whittaker and Watson, *Modern Analysis* (1927), p. 340.

then we shall have

$$f(s+\lambda)=\int_0^\infty x^{s+\lambda-1}e^{-\frac{1}{4}x^2}W_{k,m}(\tfrac{1}{2}x^2)\,dx$$

$$=\int_0^\infty x^{s-1}dx\int_0^\infty (xy)^{\frac{1}{2}}J_\nu(xy)\,y^\lambda e^{-\frac{1}{4}y^2}W_{k,m}(\tfrac{1}{2}y^2)\,dy$$

$$=\int_0^\infty y^{\lambda+\frac{1}{2}}e^{-\frac{1}{4}y^2}W_{k,m}(\tfrac{1}{2}y^2)\,dy\int_0^\infty x^{s-\frac{1}{2}}J_\nu(xy)\,dx, \qquad (3.2)$$

the change in the order of integration being permissible under certain conditions which we shall presently examine.

We know* that

$$\int_0^\infty x^q J_n(ax)\,dx=\frac{2^q}{a^{q+1}}\frac{\Gamma(\frac{1}{2}n+\frac{1}{2}q+\frac{1}{2})}{\Gamma(\frac{1}{2}n-\frac{1}{2}q+\frac{1}{2})},\ R(q)<1,$$
$$R(n+q+1)>0, \qquad (3.3)$$

and therefore from (3.1) we obtain

$$f(\lambda+s)=2^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})}\int_0^\infty y^{\lambda-s}e^{-\frac{1}{4}y^2}W_{k,m}(\tfrac{1}{2}y^2)\,dy$$

which reduces to the functional equation

$$f(\lambda+s)=2^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})}f(\lambda+1-s). \qquad (3.4)$$

4. The change in the order of integration that we have carried out above in (3.2) can be justified in the following manner:—

Suppose that

$$F(y)=y^{\lambda+\frac{1}{2}}e^{-\frac{1}{4}y^2}W_{k,m}(\tfrac{1}{2}y^2)\int_0^\infty x^{s-\frac{1}{2}}J_\nu(xy)\,dx, \qquad (4.1)$$

and

$$\phi(x)=x^{s-\frac{1}{2}}\int_0^A (xy)^{\frac{1}{2}}J_\nu(xy)\,y^\lambda e^{-\frac{1}{4}y^2}W_{k,m}(\tfrac{1}{2}y^2)\,dy, \qquad (4.2)$$

where A is small.

By the help of (3.3), we get

$$F(y)=2^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})}y^{\lambda-s}e^{-\frac{1}{4}y^2}W_{k,m}(\tfrac{1}{2}y^2),$$

where $R(s-\frac{1}{2})<1$, $R(s+\nu+\frac{1}{2})>0$; and it is easy to see that $F(y)$ is uniformly convergent for all values of y, when $R(\lambda-s)>0$.

Also when $2m$ is not an integer,

* H. Weber: *Jour. für. Math.* 69 (1869), 222-37

$$W_{k,m}(y)=\frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-m+k)}y^{\frac{1}{2}+m}e^{-\frac{1}{2}y}{}_1F_1\left(\tfrac{1}{2}+m-k\,;\,2m+1\,;\,y\right)$$
$$+\frac{\Gamma(2m)}{\Gamma(\frac{1}{2}+m-k)}y^{\frac{1}{2}-m}e^{-\frac{1}{2}y}{}_1F_1\left(\tfrac{1}{2}-m-k\,;-2m+1\,;\,y\right),$$

when $R(k\pm m-\frac{1}{2})<0$, and that for all values of ν,

$$J_\nu(y)=\sum_{r=0}^{\infty}\frac{(-)^r y^{\nu+2r}}{2^{\nu+2r}r!\,\Gamma(\nu+r+1)}.$$

Therefore $\phi(x)$ converges uniformly for all values of $x\geqslant 0$, provided $R(s+\nu)>\frac{1}{2}$ and $R(\lambda+\nu\pm 2m)>-\frac{5}{2}$. (4.3)

Again consider the integral

$$I=\int_T^\infty x^{s-1}dx\int_{T'}^\infty\left|(xy)^{\frac{1}{2}}J_\nu(xy)\,y^\lambda e^{-\frac{1}{4}y^2}\,W_{k,m}(\tfrac{1}{2}y^2)\right|dy,$$

where T and T' are as large as we please. From the asymptotic expansions* we obtain that

$$|W_{k,m}(\tfrac{1}{2}y^2)|=O(e^{-\frac{y^2}{4}}y^{2k})$$
$$|J_\nu(y)|=O(y^{-\frac{1}{2}}),$$

and hence

$I<k\int_T^\infty x^{s-1}\,dx\int_{T'}^\infty e^{-\frac{1}{2}y^2}y^{\lambda+2k}dy$, ($k$ being a constant) which can be made as small as we please, provided $R(s)<0$.

Therefore by (4.3) and (4.4), the change in the order of integration is permissible when

$$\left.\begin{array}{l}R(s)<0,\ R(s+\nu)>\frac{1}{2},\ R(\lambda-s)>0\\ R(k-\frac{1}{2}\pm m)\leqslant 0,\ R(\lambda+\nu\pm 2m)>-\frac{5}{2}.\end{array}\right\}\qquad(4.5)$$

5. Now, we substitute the values of $f(\lambda+s)$ and $f(\lambda+1-s)$ from (2.2) in (3.4) and simplify. We obtain

$$\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\lambda+m+\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\lambda-m+\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}\lambda-\tfrac{1}{2}s-k+\tfrac{8}{2})$$
$$\times\Gamma(\tfrac{1}{2}\nu-\tfrac{1}{2}s+\tfrac{3}{4})=\Gamma(m+\tfrac{1}{2}\lambda-\tfrac{1}{2}s+1)\,\Gamma(\tfrac{1}{2}\lambda-\tfrac{1}{2}s-m+1)$$
$$\times\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\lambda-k+1)\,\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4}).$$

The above relation is satisfied by taking either

$$\left.\begin{array}{l}\lambda=\nu+2m-\frac{1}{2}\\ m=-k+\frac{1}{2}\end{array}\right\}\qquad(5.1)$$

* Whittaker and Watson: *loc cit.*, pp. 342 and 368.

or,

$$\left.\begin{aligned}\lambda &= \nu-2m-\tfrac{1}{2}\\ m &= k-\tfrac{1}{2}.\end{aligned}\right\} \tag{5.2}$$

From (5.1) and (5.2) we see that

$$x^{\nu-2k+\frac{1}{2}}e^{-\frac{1}{4}x^2}W_{k,\,\pm k\mp\frac{1}{2}}(\tfrac{1}{2}x^2)$$

is R_ν, that is

$$x^{\nu-2k+\frac{1}{2}}e^{-\frac{1}{4}x^2}W_{k,\,k-\frac{1}{2}}(\tfrac{1}{2}x^2)$$
$$=\int_0^\infty (xy)^{\frac{1}{2}}\,J_\nu(xy)\,y^{\nu-2k+\frac{1}{2}}e^{-\frac{1}{4}y^2}W_{k,\,k-\frac{1}{2}}(\tfrac{1}{2}y^2)\,dy, \tag{5.3}$$

and

$$x^{\nu-2k+\frac{1}{2}}e^{-\frac{1}{4}x^2}W_{k,\,-k+\frac{1}{2}}(\tfrac{1}{2}x^2)$$
$$=\int_0^\infty (xy)^{\frac{1}{2}}J_\nu(xy)\,y^{\nu-2k+\frac{1}{2}}e^{-\frac{1}{4}y^2}W_{k,\,-k+\frac{1}{2}}(\tfrac{1}{2}y^2)\,dy. \tag{5.4}$$

6. In the same manner we have obtained*

$$f(s)=2^{\frac{1}{2}s-1}\frac{\Gamma(m+\frac{1}{2}s+\frac{1}{2})\,\Gamma(-k-\frac{1}{2}s)\,\Gamma(-m+\frac{1}{2}s+\frac{1}{2})}{\Gamma(\frac{1}{2}-k+m)\,\Gamma(-k-m+\frac{1}{2})}, \tag{6.1}$$

where $R(m-\frac{1}{2})<R(\frac{1}{2}s)<R(-k)$,

where $$f(s)=\int_0^\infty x^{s-1}e^{\frac{1}{4}x^2}\,W_{k,\,m}(\tfrac{1}{2}x^2)\,dx.$$

If now $x^\lambda e^{\frac{1}{4}x^2}W_{k,m}(\frac{1}{2}x^2)$ is R_ν, so that

$$x^\lambda e^{\frac{1}{4}x^2}W_{k,\,m}(\tfrac{1}{2}x^2)=\int_0^\infty (xy)^{\frac{1}{2}}\,J_\nu(xy)\,y^\lambda e^{\frac{1}{4}y^2}W_{k,\,m}(\tfrac{1}{2}y^2)\,dy, \tag{6.2}$$

then we shall have

$$f(s+\lambda)=\int_0^\infty x^{\lambda+s-1}e^{\frac{1}{4}x^2}W_{k,\,m}(\tfrac{1}{2}x^2)\,dx$$
$$=\int_0^\infty x^{s-1}dx\int_0^\infty (xy)^{\frac{1}{2}}\,J_\nu(xy)\,y^\lambda e^{\frac{1}{4}y^2}W_{k,\,m}(\tfrac{1}{2}y^2)\,dy$$
$$=\int_0^\infty y^{\lambda+\frac{1}{2}}e^{\frac{1}{4}y^2}W_{k,\,m}(\tfrac{1}{2}y^2)\,dy\int_0^\infty x^{s-\frac{1}{2}}J_\nu(xy)\,dx,$$

the change in the order of integration being permissible under certain conditions which can be obtained by the method given in **4**. It can further be shown that

$$f(s+\lambda)=2^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})}f(\lambda+1-s), \tag{6.3}$$

a functional equation.

* S. C. Dhar: *loc. cit.*

Now substitute in (6.3) the values of $f(\lambda+s)$ and $f(\lambda+1-s)$ from (6.1) and simplify. We obtain

$$\Gamma(\tfrac{1}{2}\nu-\tfrac{1}{2}s+\tfrac{3}{4})\,\Gamma(m+\tfrac{1}{2}\lambda+\tfrac{1}{2}s+\tfrac{1}{2})\,\Gamma(-\tfrac{1}{2}\lambda-\tfrac{1}{2}s-k)\times\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\lambda-m+\tfrac{1}{2})$$
$$=\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}s+\tfrac{1}{4})\,\Gamma(m+\tfrac{1}{2}\lambda-\tfrac{1}{2}s+1)\,\Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\lambda-k-\tfrac{1}{2})\times\Gamma(\tfrac{1}{2}\lambda-\tfrac{1}{2}s-m+1).$$

The above relation is satisfied by taking either,

$$m+\tfrac{1}{2}\lambda+\tfrac{1}{2}=\tfrac{1}{2}\nu+\tfrac{1}{4} \text{ and } \tfrac{1}{2}\lambda-m+\tfrac{1}{2}=-\tfrac{1}{2}\lambda-k-\tfrac{1}{2}$$

which give

$$\lambda=\nu-2m-\tfrac{1}{2} \text{ and } k=-\nu+3m-\tfrac{1}{2}, \tag{6.4}$$

or

$$m+\tfrac{1}{2}\lambda+\tfrac{1}{2}=-\tfrac{1}{2}\lambda-k-\tfrac{1}{2} \text{ and } \tfrac{1}{2}\lambda-m+\tfrac{1}{2}=\tfrac{1}{2}\nu+\tfrac{1}{4}$$

which give

$$\lambda=\nu+2m-\tfrac{1}{2} \text{ and } k=-\nu-3m-\tfrac{1}{2}. \tag{6.5}$$

Thus we see that

$$x^{\nu\mp 2m-\frac{1}{2}}e^{\frac{1}{4}x^2}W_{-\nu\pm 3m-\frac{1}{2},\,m}(\tfrac{1}{2}x^2) \text{ is } R_\nu. \tag{6.6}$$

ON MAXIMAL INTEGRAL FUNCTIONS*

By V. GANAPATHY IYER, *Annamalai University*.

[Received 3 May 1940]

1. The object of this paper is to discuss the properties of a special class of integral functions defined as follows. Let $g(z)$ be an integral function and ρ, d two positive numbers. We shall say that $g(z)$ *is a maximal integral function of order* ρ *and type* d *when*

$$\lim \frac{\log |g(z)|}{|z|^{\rho}} = d, \tag{1}$$

as $|z| \to \infty$ *outside a system of circles the sum of whose radii is finite*. We shall denote this fact by saying that $g(z)$ belongs to $\mathbf{M}(\rho, d)$.

1.1. It may be noted that there is an arbitrariness in the choice of the circles outside which (1) holds. For instance, if S_1 is a system of circles outside which (1) holds, S_2 any other system the sum of whose radii is finite, it is evident that we can take the totality of the circles in S_1 and S_2 as the circles, in the above definition, outside which (1) holds. But we can, for a given function, choose a system of circles which is more or less unique as follows. Firstly, the number of circles must be infinite since otherwise $g(z)$ must be a polynomial, which is impossible since ρ and d are positive. So $g(z)$ is a transcendental integral function and the number of circles is infinite. Next, we may suppose that the centres of the circles have no finite limit point; for, if z_0 is such a limit point we can replace the infinity of circles whose centres tend to z_0 by a circle round z_0 as centre and a finite number of other circles, since the sum of the radii is finite. So the circles may be supposed to be infinite in number whose centres have infinity as their sole limit point. It may be that we can suppress an infinity of these circles, and still (1) may hold outside the remaining circles. If this be so, we suppress those circles. Thus we may suppose that the circles are the least possible in the sense that an infinity of them cannot be

* An abstract of this paper was communicated to the Eleventh Conference of the Indian Mathematical Society held in December 1939 at Hyderabad (Deccan).

omitted without (1) ceasing to hold outside the remaining circles. This least system of circles associated with $g(z)$ in the above definition we shall refer to as *the exceptional circles*. We shall suppose their centres $\{u_n\}$, $n=1, 2, \ldots$, arranged according to non-decreasing moduli. Let r_n be the radius of the circle with centre u_n. Then by hypothesis Σr_n is convergent. We shall denote a circle with centre z_0 and radius λ by $C(z_0, \lambda)$. The exceptional circles consist of the system $\{C(u_n, r_n)\}$. We shall denote the system of non-overlapping domains determined by the circles $\{C(u_n, r_n)\}$ by $\{D_n\}$, $n=1, 2, \ldots$. Each domain D_n is of finite diameter, and the diameter of D_n tends to zero as $n \to \infty$.

1.2. We shall have to deal with a sub-class of $\mathbf{M}(\rho, d)$ defined as follows. Let $g(z)$ be an integral function having only simple* zeros $\{z_n\}$, $n=1, 2, \ldots$. We shall say that *$g(z)$ is a proper maximal integral function of order ρ and type d if*

(i) *$g(z)$ belongs to $\mathbf{M}(\rho, d)$;*

(ii) $\lim\limits_{n\to\infty} \dfrac{\log |g'(z_n)|}{|z_n|^\rho} = d.$

We shall denote this fact by saying that *$g(z)$ belongs to* $\mathbf{PM}(\rho, d)$.

1.3. Let $f(z)$ be an integral function and

$$M(r, f) = \max_{|z| \leqslant r} |f(z)|.$$

We say that $f(z)$ is of order ρ and type d when

$$\overline{\lim_{r\to\infty}} \frac{\log M(r, f)}{r^\rho} = d. \tag{2}$$

We shall denote by $C(\rho, d)$ *the class of all functions of order ρ and type* LESS *than d, that is, all functions $f(z)$ such that*

$$\overline{\lim_{r\to\infty}} \frac{\log M(r, f)}{r^\rho} < d. \tag{3}$$

2. We first prove the following lemmas.

LEMMA 1. *Let $\lambda > 0$. Then the ring $r < |z| < r+\lambda$ contains a circle $|z| = r'$ having no point in common with the exceptional*

* $g(z)$ can have a finite number of multiple zeros, but these can be omitted from consideration as is easily seen from the definition which follows.

circles provided $r \geqslant r_0 = r_0(\lambda)$*. *Similarly, the ring* $r < |z - z_0| < r + \lambda$ *contains such a circle if at least one of* r *or* $|z_0|$ *is sufficiently large.*

The proof follows from the fact that the sum of the radii of the exceptional circles is finite.

2.1. LEMMA 2. *A function of* $\mathbf{M}(\rho, d)$ *is of order* ρ *and type* d.

PROOF. By Lemma 1, there is a circle $|z| = r'$ in $r < |z| < r+1$ which has no point in common with the exceptional circles if r is sufficiently large. Since $r' < r+1$ and $\log M(r,f)$ is an increasing function of r, we get by (1)

$$\log M(r,f) \leqslant \log M(r', f) \leqslant (d+\varepsilon) r^{\rho} \dagger$$

if r is sufficiently large. Hence

$$\varlimsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}} \leqslant d.$$

But by (1),

$$\varlimsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}} \geqslant d.$$

So the lemma follows from these two inequalities.

2.2. The following lemma is an easy deduction from the definition of the class $\mathbf{M}(\rho, d)$.

LEMMA 3. *If* $g(z)$ *belongs to* $\mathbf{M}(\rho, d)$ *so does* $g(z) - a$ *for any finite* a. *The exceptional circles for the two functions can be taken to be the same.*

3. The first question that arises in this connection is the existence of functions of the type defined in **1**. I have shown elsewhere‡ the existence of functions belonging to $\mathbf{PM}(\rho, d)$ and indicated a method of constructing any number of such functions for a given ρ and d. An example of a function belonging to $\mathbf{M}(\rho, d)$ but not to $\mathbf{PM}(\rho, d)$ will be given towards the end. So we can state the following as an existence theorem.

* $r_0(\lambda)$ means that r_0 depends on λ. A symbol of the form $A(f, g, \ldots, \lambda, \mu \ldots)$ will mean a quantity depending on the functions $f, g, \ldots$ and parameters $\lambda, \mu, \ldots$.

† In the sequel ε will indicate a small positive number.

‡ "On Effective Sets of Points in relation to Integral Functions", *Trans. Amer. Math. Soc.* 42 (1937), 358-65, Lemma 4. Also, "Some Theorems on Functions Regular in an Angle," *Quart. Jour. Math.* (Oxford), 9 (1938), 206-15, Lemma 1.

THEOREM 1. *Given $\rho>0$ and $d>0$, any number of functions can be constructed belonging to* $\mathbf{M}(\rho, d)$ *and* $\mathbf{PM}(\rho, d)$.

3.1. THEOREM 2. *Let $g(z)$ belong to* $\mathbf{M}(\rho, d)$. *Let $\{D_n\}$ be the system of non-overlapping domains determined by the exceptional circles. Then there exists an n_0 such that for all $n \geqslant n_0$, D_n contains at least one zero of $g(z)$. Moreover, the number of zeros of $g(z)$ and $g(z)-a$ is the same in each D_n after a certain stage.**

PROOF. If the first part of the theorem were not true, there would be an infinity of domains D_n none of which will contain a zero of $g(z)$. Let $\varepsilon>0$ and D_n one of the domains containing no zero. If n is large enough we have by (1),

$$(d-\varepsilon)|z|^{\rho} \leqslant \log |g(z)| \leqslant (d+\varepsilon)|z|^{\rho}, \tag{4}$$

for z on the boundary of D_n. Since $\log |g(z)|$ is harmonic inside D_n and since a harmonic function cannot attain its maximum or minimum inside D_n, it follows that the relation

$$(d-2\varepsilon)|z|^{\rho} \leqslant \log |g(z)| \leqslant (d+2\varepsilon)|z|^{\rho} \tag{5}$$

holds inside D_n as well if n is large enough, remembering that the diameter of D_n is finite. Hence all the domains D_n containing no zero of $g(z)$ can be suppressed and still (1) will hold outside the remaining domains. But this is contrary to the definition of the exceptional circles, since suppressing an infinity of $\{D_n\}$ is equivalent to suppressing an infinity of the circles which define them. So the first part of the theorem is true. To prove the second part in the precise form stated in the foot-note, let a be such that $|a| \leqslant k$. Let N be so great that for all $n \geqslant N = N(k)$, the inequality

$$|g(z)| > k \tag{6}$$

holds on the boundary on D_n. By (1) we can find such an N. By using Rouche's theorem† and the inequality (6), we find that $g(z)$ and $g(z)-a$ have the same number of zeros in D_n for $n \geqslant N$. This completes the proof of the theorem.

3.2. We are unable to assert that every circle of the exceptional system after a certain stage must contain at least one zero.

* More precisely we can state that if a is any number such that $|a| \leqslant k$, then there exists an $N=N(k)$ such that $g(z)$ and $g(z)-a$ have the same number of zeros in D_n for $n \geqslant N$.

† See E. C. Titchmarsh, *Theory of Functions*, (Oxford), pp. 116-7. Rouche's theorem is usually proved for a simple closed contour. But it can be extended to the case of several closed contours outside one another contained inside a single closed contour. See the second proof of this theorem in the reference cited.

But we can prove what is very nearly the same thing for a special sub-sequence of the exceptional circles. An exceptional circle will be called a *boundary circle* if its circumference has at least one point* in common with the boundary of a domain of the non-overlapping system determined by the exceptional circles. We can now state

THEOREM 3. *Let $C(u_n, r_n)$ be a boundary circle whose centre u_n is at a sufficiently large distance from the origin. Let $\lambda > 0$. Then a concentric circle with radius $r_n(1+\lambda)$ will contain at least one zero of $g(z)$.*

PROOF. If the theorem were not true there would be an infinity of boundary circles such that the corresponding concentric circles contain no zero of $g(z)$ inside them. Let $C(u_n, r_n)$ be such a boundary circle and z_n a point on its circumference common to the boundary of a domain of the non-overlapping system. Let $\varepsilon > 0$. Draw the circle T with centre z_n and radius $2r_n(1+\lambda)$. The circle T obviously contains the circle with centre u_n and radius $r_n(1+\lambda)$. Let

$$h(z) = \frac{g(z)}{g(z_n)}.$$

If n is sufficiently large, $h(z)$ is an integral function since $g(z_n) \neq 0$. By (1), (2) and Lemma 2, we have, if n is large enough, for z inside or on the circumference of T,

$$\begin{aligned} \log|h(z)| &\leqslant 2\varepsilon |z_n|^\rho \\ \log|h(z_n)| &= 0 \geqslant -2\varepsilon |z_n|^\rho. \end{aligned} \tag{7}$$

3.3. At this stage we use the following result due to V. Bernstein.†

LEMMA 4. *Let $F(z)$ be regular in $|z| \leqslant R$. Let it satisfy the conditions*

$$\log|F(z)| \leqslant A, \ |z| \leqslant R$$

and

$$\log|F(0)| \geqslant -A.$$

Then to every given pair of numbers $k < 1$ and $\eta > 0$ there exists an $H = H(k, \eta)$ depending only on k and η such that

$$\log|F(z)| \geqslant -HA,$$

* Since each D_n is made up of a finite number of circles, if any one of these has one point in common with the boundary of D_n, there will be a whole arc of the circumference in common with the boundary.

† *Annali di Mat.* 12 (1934), p. 179.

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for $|z| \leqslant kR$ *except perhaps in a set of circles the sum of whose radii does not exceed* ηR.

3.4. Applying Lemma 4 to the function $h(z)$ satisfying (7) in T, we get

$$\log |h(z)| \geqslant -2\varepsilon H(k,\eta)|z_n|^\rho \tag{8}$$

in $|z-z_n| \leqslant 2kr_n(1+\lambda)$, except may be in a system of circles of total radii $2\eta r_n(1+\lambda)$. It is evident that by properly choosing k and η we can find a circle $C(u_n, r'_n)$ with $r_n < r'_n < r_n(1+\lambda)$ such that (8) holds on the circumference of this circle; it is easy to see that this choice of k and η depends only on λ. Since $g(z)$ has no zero in $C\left\{ u_n, r_n(1+\lambda) \right\}$, it follows that $\log |g(z)|$ is harmonic in $C(u_n, r'_n)$, and therefore (8) holds throughout the interior of this circle and *a fortiori* inside and on $C(u_n, r_n)$. Hence we get

$$\log |g(z)| \geqslant [d-3\varepsilon H(k, \eta)]|z|^\rho \tag{9}$$

inside and on $C(u_n, r_n)$ since the radius $r_n \to 0$. Combining (9) with Lemma 2 we conclude that we can suppress the circles $C(u_n, r_n)$ and that (1) still holds outside the remaining circles. But this contradicts the definition of the exceptional circles. Hence the theorem follows. Since $g(z)-a$ is also maximal, we can assert that $g(z)=a$ has also at least one root in the circles stated in the theorem. But we cannot state anything so precise as in the previous theorem.

3.5. THEOREM 4. *Let* $g(z)$ *belong to* $\mathbf{M}(\rho, d)$. *Let* $n(u, t)$ *denote the number of zeros of* $g(z)$ *in the circle with centre* u *and radius* t. *Let* $0 < k < 1$ *and* $|z| = r$. *Then*

$$\lim_{r\to\infty} \frac{n(z, kr)}{r^\rho} = dB(k),$$

where

$$B(k) = \frac{k}{2\pi} \frac{d}{dk} \int_0^{2\pi} |1+ke^{i\phi}|^\rho \, d\phi.$$

Also if $n(r) = n(0, r)$, *then*

$$\lim_{r\to\infty} \frac{n(r)}{r^\rho} = \rho d.$$

PROOF. Let $0 < r_1 < r_2$. By Jensen's formula* we have

$$\int_{r_1}^{r_2} \frac{n(z, t)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \left\{ \log|f(z+r_2e^{i\phi})| - \log|f(z+r_1e^{i\phi})| \right\} d\phi. \tag{10}$$

* See E. C. Titchmarsh, *l.c.*, pp. 125-6.

Let $0<k<k_1<1$. If r is large enough we can, by Lemma 1, choose t_1 and t_2, $0<t_1, t_2<1$, such that the circles with centre z and radii $kr+t_1$ and k_1r-t_2 have no point in common with the exceptional circles. Using this fact and taking $r_1=kr+t_1$, $r_2=k_1r-t_2$ in (10), we find by using (1) that*

$$\overline{\lim_{r\to\infty}}\frac{n(z,kr)}{r^\rho}\log\frac{k_1}{k}\leqslant\frac{d}{2\pi}\int_0^{2\pi}\left\{|1+k_1e^{i\phi}|^\rho-|1+ke^{i\phi}|^\rho\right\}d\phi.$$

From this, by letting $k_1\to k$ we get

$$\overline{\lim_{z\to\infty}}\frac{n(z,kr)}{r^\rho}\leqslant dB(k). \tag{11}$$

Similarly, taking $0<k_1<k<1$ and proceeding as above, we conclude that

$$\underline{\lim_{r\to\infty}}\frac{n(z,kr)}{r^\rho}\geqslant dB(k). \tag{12}$$

Combining (11) and (12) we get the first part of the theorem. The second part follows similarly by taking $z=0$ in (10) and using (1) as in the above argument. It may be noted that $B(k)$ has the following expansion:

$$B(k)=\sum_{n=1}^{\infty}2nc_n^2(\rho/2)k^{2n}, \tag{13}$$

where $c_n(\rho/2)$ is the coefficient of t^n in $(1+t)^{\rho/2}$. It is seen that $B(k)\to0$ as $k\to0$ and that $B(k)$ steadily increases with k in $0<k<1$. Also $B(k)$ tends to a finite limit as $k\to1$.

3.6. THEOREM 5. *Let $g(z)$ belong to $\mathbf{M}(\rho,d)$ and let $n(r,\alpha,\beta)$ denote the number of zeros of $g(z)$ in the sector $|z|\leqslant r$, $\alpha\leqslant\theta\leqslant\beta$, $z=re^{i\theta}$. Then*

$$\underline{\lim_{r\to\infty}}\frac{n(r,\alpha,\beta)}{r^\rho}\geqslant dc(\delta),$$

where $c(\delta)$ is a constant depending only on $\delta=\beta-\alpha$.

* If $z=re^{i\theta}$, I have used the fact that

$$\int_0^{2\pi}|e^{i\theta}+ke^{i\phi}|^\rho d\phi=\int_0^{2\pi}|1+ke^{i(\phi-\theta)}|d\phi=\int_0^{2\pi}|1+ke^{i\phi}|d\phi,$$

since $e^{i\phi}$ is of period 2π in ϕ.

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PROOF. If $\delta = 2\gamma = \beta - \alpha$, and $|z| = r$, the circle with centre at

$$z_0 = \frac{z}{1+\sin\gamma}$$

and radius $|z_0| \sin\gamma$ is inscribed in the sector $|z| \leqslant r$ and $\alpha \leqslant \theta \leqslant \beta$, z being taken on the mid-point of the arc of the sector. So by the previous theorem

$$\varliminf_{r\to\infty} \frac{n(r,\alpha,\beta)}{r^\rho} \geqslant \varliminf_{r\to\infty} \frac{n(z_0, |z_0|\sin\gamma)}{r^\rho}$$

$$\geqslant \frac{d}{(1+\sin\gamma)^\rho} B(\sin\gamma).$$

Hence the theorem follows if we take

$$c(\delta) = \frac{1}{(1+\sin\delta/2)^\rho} B\left(\sin\frac{\delta}{2}\right).$$

3.7. It is proved later that*

$$\lim_{r\to\infty} \frac{n(r,\alpha,\beta)}{r^\rho}$$

exists, its value being equal to $\frac{\delta}{2\pi}\rho A$.

4. In this and in the next section, we shall be concerned with functions of the class $\mathbf{PM}(\rho, d)$. A function of this class has only simple zeros and, in addition to (1), satisfies the condition (ii) of § 1.2. In discussing the properties of these functions we shall have to make frequent use of the following result.†

LEMMA 5. *Let* $\delta > 0$, $0 \leqslant d_1 < d_2$. *Let* $z = re^{i\theta}$, *and* θ_0 *a point in* $0 \leqslant \theta \leqslant 2\pi$. *Then, corresponding to every* $\rho > 0$ *there exists an integral function* $h_\rho(z)$ *of the class* $C(\rho, d_2)$ [see § 1.3] *satisfying the condition*

$$\varliminf \frac{\log |h_\rho(z)|}{|z|^\rho} > d_1$$

as $|z| \to \infty$ *in the angle* $|\theta - \theta_0| \leqslant \delta$.

* Vide the additional note at the end of this paper.

† A function as contemplated in Lemma 5 is easily constructed for a given $\rho > 0$ by using the well-known properties of the following functions:—

$$Q_\rho(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n^{1/\rho}}\right), \ 0 < \rho < \frac{1}{2} \ ; \ Q_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\frac{n}{\rho})}, \ \rho \geqslant \frac{1}{2}.$$

4.1. THEOREM 6. *Let $g(z)$ be a function of order ρ and type d having only simple zeros $\{z_n\}$, $n=1,2,\ldots$. Then a necessary and sufficient condition that $g(z)$ belongs to* $\mathbf{PM}(\rho, d)$ *is that the interpolation formula*

$$\frac{f(z)}{g(z)} = \sum_{n=1}^{\infty} \frac{f(z_n)}{g'(z_n)} \frac{1}{z-z_n} \tag{14}$$

holds for every function $f(z)$ of $C(\rho, d)$, the series in (14) *converging for at least one value of $z \neq z_n (n=1, 2, \ldots)$ for each function $f(z)$.*

NOTE. In proving the condition to be necessary, we can show that the series (14) converges absolutely and uniformly for all finite z different from $\{z_n\}$. In proving sufficiency, it is enough to assume the convergence of the series for one finite $z \neq z_n$ for each $f(z)$ of $C(\rho, d)$. Having proved that $g(z)$ belongs to $\mathbf{PM}(\rho, d)$ from this hypothesis, it will follow that (14) converges absolutely and uniformly for all finite z different from $\{z_n\}$.

PROOF. We first prove the necessity of the condition. By Lemma 1, we can find a sequence of circles $|z| = R_n$, $0 < R_1 < R_2 \ldots < R_n \to \infty$ as $n \to \infty$, none of which cuts any of the exceptional circles. Consider the integral

$$I_n = \frac{1}{2\pi i} \int_{|x|=R_n} \frac{f(x)}{g(x)} \frac{dx}{x-z}.$$

By (1), $I_n \to 0$ as $n \to \infty$ if $f(z)$ belongs to $C(\rho, d)$. The relation (14) follows by the residue theorem applied to I_n remembering that the series in (14) converges absolutely and uniformly for all finite z different from $\{z_n\}$ in virtue of the condition (ii) of § 1.2.

To prove sufficiency, we first show that

$$\lim_{n\to\infty} \frac{\log |g'(z_n)|}{|z_n|^{\rho}} = d. \tag{15}$$

If (15) is not true, there will be a sub-sequence $\{v_n\}$ of $\{z_n\}$ such that

$$\lim_{n\to\infty} \frac{\log |g'(v_n)|}{|v_n|^{\rho}} = \beta < d, \tag{16}$$

since $g(z)$ is of order ρ and type d. We may also suppose that the amplitudes of $\{v_n\}$ all tend to a definite limit θ_0 since otherwise

we can replace $\{v_n\}$ by a properly chosen sub-sequence. Let $\beta < \beta' < d$. Choose an angle $|\theta - \theta_0| \leqslant \delta$. With $\beta' = d_1$ and $d = d_2$ in Lemma 5, consider a function $h_\rho(z)$ as contemplated in the lemma. Taking $f(z) = h_\rho(z)$ in (14) and noting that the series converges for at least one finite z different from $\{z_n\}$, we easily deduce that

$$\varliminf_{n\to\infty} \frac{\log |g'(v_n)|}{|v_n|^\rho} \geqslant \varliminf_{n\to\infty} \frac{\log |h_\rho(v_n)|}{|v_n|^\rho} \geqslant \beta',$$

since all v_n after a certain stage lie in the angle $|\theta - \theta_0| \leqslant \delta$. Since $\beta' > \beta$, the last result contradicts (16). Hence (15) holds. To show that (1) holds outside a system of circles of total finite radii, we note that (14) converges absolutely and uniformly for all finite z different from $\{z_n\}$ for all $f(z)$ of $C(\rho, d)$, since (15) is true. Draw the circles with centres at z_n and radii $|z_n|^{-h}$ with $h > \rho$. Then $\sum |z_n|^{-h}$ converges. We shall show that (1) holds outside these circles. If not, there will be a sequence of points $\{v_n\}$ outside these circles whose amplitudes tend to a limit θ_0 such that

$$\lim_{n\to\infty} \frac{\log |g(v_n)|}{|v_n|^\rho} = \gamma < d. \tag{17}$$

Taking $\gamma < \gamma' < d$ and $\gamma' = d_1$, $d = d_2$, in Lemma 5 we consider the function $h_\rho(z) = f(z)$ in (14). We get

$$\left| \frac{h_\rho(z)}{g(z)} \right| \leqslant \sum_{n=1}^{\infty} \frac{|z_n|^h |h_\rho(z_n)|}{|g'(z_n)|}$$

for z outside the above circles. Since $h_\rho(z)$ belongs to $C_\rho(z)$ it follows that as $|z| \to \infty$ outside the circles considered

$$\varliminf \frac{\log |g(z)|}{|z|^\rho} \geqslant \varliminf \frac{\log |h_\rho(z)|}{|z|^\rho}.$$

In particular

$$\varliminf_{n\to\infty} \frac{\log |g(v_n)|}{|v_n|^\rho} \geqslant \varliminf_{n\to\infty} \frac{\log |h_\rho(v_n)|}{|v_n|^\rho} > \gamma', \tag{18}$$

since, as before, all v_n after a certain stage lie in $|\theta - \theta_0| \leqslant \delta$. Since $\gamma' > \gamma$, (18) contradicts (17). So the theorem follows.

4.2. THEOREM 7. *The condition* (14) *is equivalent to the two conditions*

$$(i) \ \lim_{n\to\infty} \frac{\log |g'(z_n)|}{|z_n|^\rho} = d$$

$$(ii)\ \frac{z^m}{g(z)} = \sum_{n=1}^{\infty} \frac{z_n{}^m}{g'(z_n)} \frac{1}{z-z_n},$$

where $m \geqslant 0$ *is an integer.*

PROOF. Suppose (i) and (ii) of Theorem 7 hold. Let $f(z) = c_0 + c_1 z + c_2 z^2 + \dots$ be a function of $C(\rho,d)$. Then

$$\sum_{m=0}^{\infty} |c_m|\ |z|^m \leqslant A \exp\left(\beta|z|^\rho\right),\ \beta < d.$$

Hence the double series

$$\sum_{(n,m)} \left| \frac{c_m z_n{}^m}{g'(z_n)} \cdot \frac{1}{z-z_n} \right|$$

converges for all finite z different from $\{ z_n \}$. Hence using (ii) and summing the double series

$$\sum_{(n,m)} \frac{c_m z_n{}^m}{g'(z_n)} \frac{1}{z-z_n}$$

by rows and columns, we see that (14) holds for every function of $C(\rho,d)$. On the other hand if (14) holds for every $f(z)$ of $C(\rho,d)$, (ii) of Theorem 7 obviously holds. That (i) also holds has been proved in the course of the proof of Theorem 6. This completes the proof of Theorem 7.

4.3. THEOREM 8. *The condition* (14) *of Theorem* 6 *is equivalent to the condition* (*i*) *of Theorem 7 along with the additional relation*

$$\overline{\lim_{r\to\infty}} \frac{\log m(r,g)}{\log r} = \infty, \tag{19}$$

where

$$m(r,g) = \min_{|z|=r} |g(z)|.$$

PROOF. If condition (i) of Theorem 7 and (19) hold, we note that the series

$$\sum_{1}^{\infty} \frac{z_n{}^m}{g'(z_n)} \cdot \frac{1}{z-z_n}$$

converges absolutely and uniformly for all finite z except $\{ z_n \}$, while there exists a sequence

$$R_1 < R_2 < \dots < R_n \to \infty$$

such that

$$\frac{1}{2\pi i}\int_{|x|=R_n} \frac{x^m}{g(x)} \frac{dx}{x-z} \to 0$$

as $n \to \infty$. These two together show that (ii) of Theorem 7 holds. Hence (14) holds by Theorem 7. Conversely if (14) holds, (i) of Theorem 7 holds while (19) follows from (1) and the fact that $g(z)$ belongs to $\mathbf{PM}(\rho, d)$.

4.4. The relation (19) is true for every function of order less than half. So, as an application of Theorem 8 we can state the following

THEOREM 9. *Let $g(z)$ be an integral function of order $\rho < \frac{1}{2}$ and type d. Let a be such that all the zeros of $g(z)-a$ except may be a finite number are simple. Let $\{z_n\}$, $n = 1, 2, \ldots$, be these simple zeros. Then either*

$$\lim_{n\to\infty} \frac{\log |g'(z_n)|}{|z_n|^\rho} < d,$$

or else $g(z)-a$ belongs to $\mathbf{PM}(\rho,d)$. Moreover, if there exists one value of a such that $g(z)-a$ belongs to $\mathbf{PM}(\rho, d)$, then for every value of a such that $g(z)-a$ has only a finite number of multiple zeros, $g(z)-a$ belongs to $\mathbf{PM}(\rho, d)$ and for all a, $g(z)-a$ belongs to $\mathbf{M}(\rho, d)$.

5. In this section we proceed to discuss another type of necessary and sufficient condition for functions of $\mathbf{PM}(\rho, d)$.

THEOREM 10. *Let $g(z)$ be a function of order ρ and type d and have only simple zeros $\{z_n\}$. A necessary and sufficient condition that $g(z)$ belongs to $\mathbf{PM}(\rho, d)$ is*

$$\sum_{n=1}^{\infty} \frac{f(z_n)}{g'(z_n)} = 0, \qquad (20)$$

for every function $f(z)$ of $C(\rho, d)$. The condition (20) can be replaced by the equivalent conditions

$$(i) \lim_{n\to\infty} \frac{\log |g'(z)|}{|z_n|^\rho} = d$$

and $$(ii) \sum_{n=1}^{\infty} \frac{z_n^m}{g'(z_n)} = 0, \ m \geqslant 0 \text{ being an integer.}$$

PROOF. To prove that (20) is necessary we note that if $f(z)$ belongs to $C(\rho, d)$, so does $zf(z)$. Applying (14), we get the relations

$$\frac{f(z)}{g(z)} = \sum_{n=1}^{\infty} \frac{f(z_n)}{g'(z_n)} \frac{1}{z-z_n}$$

$$\frac{zf(z)}{g(z)} = \sum_{n=1}^{\infty} \frac{z_n f(z_n)}{g'(z_n)} \frac{1}{z-z_n}. \tag{21}$$

Multiplying the first relation in (21) by z and subtracting from the second we have

$$0 = -\sum_{n=1}^{\infty} \frac{f(z_n)}{g'(z_n)} \cdot \frac{z-z_n}{z-z_n}$$

which gives (20). To prove that (20) is sufficient we first prove as in Theorem 6 that the convergence of the series in (20) implies that

$$\lim_{n\to\infty} \frac{\log |g'(z_n)|}{|z_n|^{\rho}} = d. \tag{22}$$

Next let $H(z)$ be the integral function defined by

$$\frac{H(z)}{g(z)} = \sum_{n=1}^{\infty} \frac{1}{g'(z_n)} \frac{1}{z-z_n},$$

which exists in virtue of (22). By taking $f(z) = z^m$, $m = 0, 1, 2, \ldots$ in (20), we find that

$$\frac{z^m H(z)}{g(z)} = \sum_{n=1}^{\infty} \frac{z^m}{g'(z_n)} \frac{1}{z-z_n}$$

$$= \sum_{n=1}^{\infty} \frac{z_n^m}{g'(z_n)} \frac{1}{z-z_n} + \sum_{n=1}^{\infty} \frac{1}{g'(z_n)} \frac{z^m - z_n^m}{z-z_n},$$

and since

$$\frac{z^m - z_n^m}{z-z_n} = z^{m-1} + z^{m-2} z_n + \ldots + z_n^{m-1},$$

we get by (20)

$$z^m \frac{H(z)}{g(z)} = \sum_{n=1}^{\infty} \frac{z_n^m}{g'(z_n)} \frac{1}{z-z_n}. \tag{23}$$

Combining (22) and (23) we conclude as in Theorem 7 that

$$\frac{f(z)\, H(z)}{g(z)} = \sum_{n=1}^{\infty} \frac{f(z_n)}{g'(z_n)} \frac{1}{z-z_n}. \tag{24}$$

Let $0 < d' < d$. By Theorem 1 there is a function $f(z)$ belonging to $\mathbf{M}(\rho, d')$. Taking this $f(z)$ in (24) and arguing as in Lemma 2 we find that

$$\overline{\lim_{r\to\infty}} \frac{\log M(r, H)}{r^{\rho}} \leqslant d-d'.$$

Since $d-d'$ can be as small as we please we find that

$$\overline{\lim_{r\to\infty}} \frac{\log M(r, H)}{r^{\rho}} \leqslant 0.$$

Therefore the same is true of $\left\{ H(z) \right\}^{p}$, where $p \geqslant 0$ is an integer. Hence we can take $f(z)$ to be any power of $H(z)$ in (24). Since $H(z_n) = 1$, $n = 1, 2, \ldots$ we get from (24)

$$\frac{[H(z)]^2}{g(z)} = \sum_{n=1}^{\infty} \frac{1}{g'(z_n)} \frac{1}{z-z_n} = \frac{H(z)}{g(z)}.$$

Thus we get $H(z) \equiv 0$ or $H(z) \equiv 1$ since $H(z)$ is an integral function. But $H(z_n) = 1$, $n = 1, 2, \ldots$. Hence $H(z) \equiv 1$ in (24), which then reduces to (14). So this theorem follows from Theorem 7. The equivalence of (20) with (i) and (ii) is proved in the same manner as Theorem 7 is proved from Theorem 6.

6. In certain cases it is possible to assert that a function of $\mathbf{M}(\rho, d)$ belongs to $\mathbf{PM}(\rho, d)$. The first essential condition is that the zeros of the function are all simple but for a finite number. We shall prove the following result.

THEOREM 11. *Let $g(z)$ belong to $\mathbf{M}(\rho, d)$ and have only simple zeros $\left\{ z_n \right\}$. Let $\mu_n = \min_{m \neq n} |z_m - z_n|$. Let $\left\{ D_n \right\}$ be the system of non-overlapping domains determined by the exceptional circles $\left\{ C(u_n, r_n) \right\}$. Then $g(z)$ will belong to $\mathbf{PM}(\rho, d)$ if any one of the following conditions holds:*

(*i*) *There exists a fixed $\lambda > 0$ such that* $n(z, \lambda) = o\left[\frac{|z|^{\rho}}{\log |z|}\right]$*

* It can be shown that $n(z, t) = o(|z|^{\rho})$ if $t = o(|z|)$. Hence whatever be λ, finite and positive, $n(z, \lambda) = o(|z|^{\rho})$. But the condition stated requires something more and so has to be postulated.

as $|z| \to \infty$ *and* $\overline{\lim_{n\to\infty}} \frac{\log 1/\mu_n}{\log |z_n|} < \infty$;

(*ii*) $n(z, \lambda) < \infty$ *and* $\overline{\lim_{n\to\infty}} \frac{\log 1/\mu_n}{|z_n|^\rho} \leqslant 0$;

(*iii*) $n(D_n) < \infty$, $\overline{\lim_{n\to\infty}} \frac{\log 1/\mu_n}{|z_n|^\rho} \leqslant 0$

and $\underline{\lim_{n\to\infty}} \frac{\log r_n}{|z_n|^\rho} \geqslant 0$, *where* $n(D_n)$ *denotes the number of zeros of* $g(z)$ *in* D_n.

PROOF. We shall give a proof of (i). The remaining cases can be dealt with similarly. What we have to show is that (ii) of § 1.2 holds. Let z_n be a zero of $g(z)$ such that $|z_n|$ is sufficiently large. By Lemma 1 we can draw a circle $C(z_n, \lambda')$, $0 < \lambda' < \lambda$ whose circumference does not cut any of the exceptional circles. Let

$$g(z) = P_n(z)Q_n(z),$$

where

$$P_n(z) = \Pi\left(1 - \frac{z}{z_\nu}\right), \tag{25}$$

the product being taken over all z_ν lying in $C(z_n, \lambda')$. We have, if n is large enough, for z in $C(z_n, \lambda')$,

$$\log |P_n(z)| \leqslant n(z_n, \lambda') \log 3. \tag{26}$$

Also for z on the circumference of $C(z_n, \lambda')$,

$$\log |P_n(z)| \geqslant n(z_n, \lambda') \log \lambda' - n(z_n, \lambda') \log |z_n - \lambda'|. \tag{27}$$

Using (26), (27) and the condition imposed on $n(z_n, \lambda')$, we easily see that

$$\log |P_n(z)| = o(|z|^\rho), \tag{28}$$

for z on the circumference of $C(z_n, \lambda')$. From (1) and (28) we deduce that

$$\log |Q_n(z)| = [d + o(1)]|z|^\rho, \tag{29}$$

for z on the circumference of $C(z_n, \lambda')$. But $Q_n(z)$ has no zeros inside $C(z_n, \lambda')$ so that $\log |Q_n(z)|$ is harmonic in the circle. Therefore it is easy to see that (29) holds in the interior of $C(z_n, \lambda')$ as well. In particular,

$$\log |Q_n(z_n)| = [d + o(1)]|z_n|^\rho, \tag{30}$$

as $n \rightarrow \infty$. But

$$g'(z_n) = P'_n(z_n) Q_n(z_n). \tag{31}$$

But we have, as before,

$$\log |P'_n(z_n)| \leqslant [n(z_n, \lambda') - 1] \log 3 - \log |z_n - \lambda'|,$$

while

$$\log |P'_n(z_n)| \geqslant [n(z_n, \lambda') - 1] \log \mu_n - n(z_n, \lambda') \log |z_n|.$$

From these two, and the conditions of $n(z_n, \lambda)$ and μ_n, we see that

$$\log |P'_n(z_n)| = o(|z_n|^\rho). \tag{32}$$

The relations (30), (31) and (32) show that (ii) of § 1.2 holds.

6.1. In the case of functions of $\mathbf{PM}(\rho, d)$ we can choose the exceptional circles so as to have their centres at the zeros of the function. In this connection we state the following theorem the proof of which is on lines similar to that of the sufficiency part of Theorem 6.

THEOREM 12. *Let $g(z)$ belong to $\mathbf{PM}(\rho, d)$ and let its zeros be $\{z_n\}$. Then the exceptional circles of $g(z)$ can be chosen to be the system $\{C(z_n, r_n)\}$ having their centres at $\{z_n\}$, and their radii r_n can be chosen to satisfy the conditions (i) Σr_n converges, and (ii) $\overline{\lim\limits_{n\to\infty}} \dfrac{1}{|z_n|^\rho} \log \dfrac{1}{r_n} \leqslant 0$. In particular we can take $r_n = B|z_n|^{-h}$ where $h > \rho$, and $B > 0$ is finite and fixed.*

6.2. The following result is an easy deduction from Theorems 2, 11 and 12 [*cf.* Theorem 9].

THEOREM 13. *Let $g(z)$ belong to $\mathbf{PM}(\rho, d)$ and let the exceptional circles be chosen as in Theorem 12. If these circles are non-overlapping after a certain stage for a proper choice of the radii r_n subject to Theorem 12, then $g(z) - a$ also belongs to $\mathbf{PM}(\rho, d)$.*

7. We can state the following for the class of functions $\mathbf{M}(\rho, d)$.

THEOREM 14. *Let $g(z)$ belong to $\mathbf{M}(\rho, d)$. Let $h(z)$ be a function of order less than ρ. Then $g(z)+h(z)$ and $g(z)h(z)$ belong to $\mathbf{M}(\rho, d)$. If $g_1(z)$ belongs to $\mathbf{M}(\rho, d_1)$ and $g_2(z)$ to $\mathbf{M}(\rho, d_2)$, $g_1(z)g_2(z)$ belongs to $\mathbf{M}(\rho, d_1+d_2)$. If $g_1(z)$ and $g_2(z)$ both belong to $\mathbf{M}(\rho, d)$ and have the same zeros, then $g_1(z) = e^{P(z)} g_2(z)$, where $P(z)$ is a polynomial of degree less than ρ.*

PROOF. To prove the first two parts we have only to note* that if $f(z)$ is a function of order ρ, $\{z_n\}$ its zeros, then

$$\log |f(z)| = O(|z|^{\rho'}),\ \rho' > \rho$$

as $|z| \to \infty$ outside the circles with centres at z_n and radii $|z_n|^{-h}$, $h > \rho$, so that $\Sigma\, |z_n|^{-h}$ converges. The third part easily follows from definition. To prove the last part, if $g_1(z)$ and $g_2(z)$ have the same zeros, then

$$g_1(z) = e^{P(z)} g_2(z),$$

where $P(z)$ is a polynomial of degree *not exceeding* ρ. But if $g_1(z)$ and $g_2(z)$ both belong to $\mathbf{M}(\rho, d)$ we get

$$\log \left| \frac{g_1(z)}{g_2(z)} \right| = o(|z|^{\rho})$$

as $|z| \to \infty$ outside the exceptional circles of $g_1(z)$ and $g_2(z)$. The last result shows that $P(z)$ cannot be of degree ρ. This proves the last part.

7.1. We conclude with a few examples and some general remarks. The simplest classical example of a function of the type discussed in this paper is the canonical product associated with the lattice points $z = m + in$, $(m,\ n = 0, \pm 1, \pm 2, \dots)$, namely,

$$\sigma(z) = z\, \Pi \left[1 - \frac{z^4}{(m+in)^4} \right],$$

where m, n take all non-negative integral values except the combination $m = n = 0$. This function belongs to $\mathbf{PM}(2, \pi/2)$ as is easily seen by using the quasi-periodic† properties of $\sigma(z)$. I have shown elsewhere that the function‡

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^n}{n^{\alpha n}} \right)$$

belongs to $\mathbf{PM}\left(\frac{2}{\alpha}, \frac{\alpha}{4} \right)$. The function $g(\eta z)$ belongs to $\mathbf{PM}(\rho, d)$ if $\rho = 2/\alpha$ and $d = \frac{1}{4}\eta^{\rho}\alpha$. The same method can be employed to show that

$$\prod_{n=1}^{\infty} \left\{ 1 - \frac{z^n}{(n+r_n)^{\alpha n}} \right\},\quad r_n = \exp(-n^{\lambda}),\ \lambda > 0$$

* See E. C. Titchmarsh, *l.c.*, p. 273.

† See Whittaker and Watson, *Modern Analysis* (1920), p. 464.

‡ *Trans. Amer. Math. Soc. l.c.* Lemma 4.

also belongs to $\mathbf{PM}\left(\frac{2}{\alpha}, \frac{\alpha}{4}\right)$. So the function

$$\prod_{n=1}^{\infty}\left\{1-\frac{z^n}{n^{\alpha n}}\right\}\left\{1-\frac{z^n}{(n+r_n)^{\alpha n}}\right\} \tag{33}$$

belongs to $\mathbf{M}\left(\frac{2}{\alpha}, \frac{\alpha}{2}\right)$, but by choosing λ sufficiently great we can show that this function does not belong to $\mathbf{PM}\left(\frac{2}{\alpha}, \frac{\alpha}{2}\right)$. Similarly the function

$$\prod_{n=1}^{\infty}\left(1-\frac{z^n}{n^{\alpha n}}\right)^n \tag{34}$$

belongs to $\mathbf{M}\left(\frac{3}{\alpha}, \frac{\alpha}{9}\right)$ but not to $\mathbf{PM}\left(\frac{3}{\alpha}, \frac{\alpha}{9}\right)$, since all its zeros except one are multiple zeros. So a function of $\mathbf{M}(\rho, d)$ need not have all its zeros simple, nor does the fact that all the zeros of a function of $\mathbf{M}(\rho, d)$ are simple imply that the function belongs to $\mathbf{PM}(\rho, d)$. [For a general method of constructing functions of $\mathbf{PM}(\rho, d)$, see the references quoted in connection with Theorem 1.]

7.2. It is the property indicated by the relation (14) in Theorem 6 that suggested the name "*maximal integral functions*" for functions of the type discussed in this paper. It is known that for integral functions possessing special properties there are "flat regions" where $|f(z)|$ is comparable with $\mathbf{M}(r, f)$.* For the functions discussed in this paper, $|f(z)|$ is as large as the maximum modulus for all z except in a system of circles the sum of whose radii is finite. We cannot expect more from a transcendental integral function. Again, from Theorem 6, a function of $\mathbf{PM}(\rho, d)$ can serve as an interpolation function for *all* functions of $C(\rho, d)$ and any such function belongs to $\mathbf{PM}(\rho, d)$. The first property is possessed to the maximum possible extent by functions of $\mathbf{M}(\rho, d)$ and both properties by the functions of $\mathbf{PM}(\rho, d)$, and hence the designation adopted in this paper for such functions.

Additional Note.†

8. We shall prove that $\lim_{r\to\infty} \frac{n(r,\alpha,\beta)}{r^\rho}$ exists. For this purpose we require some preliminary notions and a theorem proved by A. Pfluger.

* See J.M. Whittaker, *Interpolatory Function Theory*, (Cambridge Tract No. 33), pp. 73-83 and the references quoted there.

† Added 27th June 1940.

Let E be a set of points on the positive real axis $0 \leqslant r < \infty$. Let $E(r)$ be the measure of the component of E in $(0, r)$. If $\lim_{r\to\infty} E(r)/r = 1$, we say that E is of linear density one. Let $f(z)$ be regular in the sector $\alpha < \theta < \beta$ and let it be of order ρ and finite type in this domain. Let

$$h(\theta) = \overline{\lim_{r\to\infty}} \frac{\log |f(re^{i\theta})|}{r^\rho}.$$

The function $f(z)$ is said to be of regular asymptotic behaviour in the sector, if for every ϕ in $\alpha < \phi < \beta$ there exists a set $E(\phi)$ of linear density one on the line $\theta = \phi$ such that

$$\lim \frac{\log |f(re^{i\phi})|}{r^\rho} = h(\phi),$$

as $r \to \infty$ over $E(\phi)$. For this class of functions the following general result has been proved by A. Pfluger:*

THEOREM A. *Let $f(z)$ be regular of order ρ and finite type in $\alpha < \theta < \beta$. Let it be of regular asymptotic behaviour in this sector. Let θ_1 and θ_2 be two points in $\alpha < \theta < \beta$ where $h'(\theta)$ is continuous. Then*

$$2\pi \lim_{r\to\infty} \frac{n(r,\theta_1,\theta_2)}{r^\rho} = \rho \int_{\theta_1}^{\theta_2} h(\theta)\,d\theta + \frac{1}{\rho}[h'(\theta_2) - h'(\theta_1)].$$

8.1. We can now prove

THEOREM 15. *Let $f(z)$ be a function of* $\mathbf{M}(\rho,d)$. *Then*

$$\underline{\lim_{r\to\infty}} \frac{n(r,\alpha,\beta)}{r^\rho} = \frac{d\rho}{2\pi}(\beta - \alpha).$$

PROOF. First we note that a function $f(z)$ of $\mathbf{M}(\rho, d)$ is of regular asymptotic behaviour for every θ in $0 \leqslant \theta \leqslant 2\pi$. Moreover $h(\theta) \equiv d$ for all θ in $0 \leqslant \theta \leqslant 2\pi$ and $h'(\theta) \equiv 0$. Hence applying Pfluger's Theorem A we get the required result.

* *Commentarii Mathematici Helvetici*, 12, 1, p. 54, equation (3.20). The proof of the Theorem A even when adapted for functions of $\mathbf{M}(\rho, d)$ is long, and so only the result is quoted.

A SET OF AXIOMS FOR TOPOLOGICAL ALGEBRA*

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1. *Introduction.* Set Theoretic Topology is usually based on a set of axioms involving an undefined concept—the closure of a set. They are the following four axioms of Kuratowski: †

$$\left.\begin{array}{ll} K_1. & \overline{A+B} = \bar{A}+\bar{B} \\ K_2. & \bar{A} \supset A \\ K_3. & \bar{\bar{A}} = \bar{A} \\ K_4. & \bar{O} = O \end{array}\right\}. \qquad (K)$$

The junction operation between two sets A and B which is indicated by AJB has been defined by Unaster and Kuratowski‡ as $A\bar{B}+B\bar{A}$. It is possible to replace the above set of four axioms involving the closure functions by a set of axioms involving the concept of junctional operation. In this paper I propose to introduce a set of five axioms relating to the junction of two sets as a primitive binary operation and then show that they are equivalent to Kuratowski's axioms when AJB, $\bar{A}$ and $\bar{B}$ are connected by the equation $AJB = A\bar{B}+B\bar{A}$. The five junction operation axioms are

$$\left.\begin{array}{ll} J_1. & AJB = BJA \\ J_2. & AJA = A \\ J_3. & AJO = O \\ J_4. & AJ(B+C) = AJB+AJC \\ J_5. & AJ(AJB) = AJB \end{array}\right\}. \qquad (J)$$

We shall now establish the equivalence of the axioms.

* I wish to express my thanks to Dr. R. Vaidyanathaswamy for the help he gave in the preparation of this paper.

† Sur l' operation de l' Analysis Situs, *Fundamenta Mathematicae*, Tome 3, or Kuratowski, *Topologie* I.

‡ Sur les ensembles Connexes, *Fundamenta Mathematicae*, Tome 2.

2. *To prove that the set of axioms (J) imply the set of axioms (K) if $AJB = A\bar{B} + B\bar{A}$.*

We have by J_2,

$$A = AJA = A\bar{A} + A\bar{A} \text{ (by definition)} = A\bar{A}.$$

Hence $$A \subset \bar{A} \text{ which is } K_2. \tag{1}$$

When $A = 1$, it follows from (1) that $1 \subset \bar{1}$,

and hence $$1 = \bar{1}, \tag{2}$$

and $$1JA = 1.\bar{A} + A.\bar{1} = \bar{A} + A = \bar{A}, \tag{3}$$

by (1) and (2).

J_3 gives $$AJ(B+C) = AJB + AJC.$$

If $A = 1$ then

$$1J(B+C) = 1JB + 1JC.$$

But $$1J(B+C) = \overline{B+C} \text{ by (3)};$$

hence $$\overline{B+C} = \bar{B} + \bar{C}, \text{ which is } K_1. \tag{4}$$

From J_5, we have when $A = 1$,

$$1J(1JB) = 1JB.$$

Hence by (3), $$\bar{\bar{B}} = \bar{B}, \text{ which is } K_3. \tag{5}$$

We have from J_5 when $A = 1$, $1JO = O$,

and hence by (3), $$\bar{O} = O, \text{ which is } K_4.$$

3. *Next we shall prove that the set of axioms (K) imply the set of axioms (J) under the hypothesis $AJB = A\bar{B} + B\bar{A}$.*

From the symmetry of the definition of AJB we see that $AJB = BJA$, which is J_1.

$$AJA = A\bar{A} + A\bar{A} \text{ (by definition)}$$
$$= A\bar{A} = A \text{ (by } K_2\text{), which is } J_2.$$

By definition $AJO = A\bar{O} + O\bar{A} = O$ (by K_4), which is J_3.

$$AJ(B+C) = A\,\overline{B+C} + (B+C)\bar{A}$$
$$= A\bar{B} + A\bar{C} + B\bar{A} + C\bar{A} \text{ (from } K_1\text{)}$$
$$= AJB + BJC, \text{ which is } J_4.$$

$$AJ\,AJB = AJ(A\bar{B} + \bar{A}B)$$
$$= A(\overline{A\bar{B} + B\bar{A}}) + \bar{A}(A\bar{B} + \bar{A}B)$$

by definition

$$= A\bar{B} + B\bar{A} + \text{a quantity contained in } A\bar{B} \text{ (from axioms } K_2 \text{ and } K_3).$$

$$= A\bar{B} + B\bar{A} = AJB, \text{ which is } J_5.$$

It is necessary to make a remark here which will be of use in the proof of the independence of the axioms J_4 and J_5 from the rest. It may be easily seen from the above proof that axioms K_1, K_2 and $K_4 \supset J_1, J_2, J_3$ and J_4 while K_2, K_3 and $K_4 \supset J_1, J_2, J_3$ and J_5. Therefore a space satisfying the axioms K_1, K_2 and K_4 satisfies all the axioms of the set (J) except J_5, and similarly all the axioms of the set (J) except J_4 are satisfied in a space in which only K_2, K_3 and K_4 are true.

4. We shall establish the independence of the set of axioms in (J). For this we have to construct examples which satisfy only four of the five conditions expressed in the postulate (J).

EXAMPLE 1. Define AJB as B. Obviously $AJB \neq BJA$ and hence J_1 is not satisfied.

$$AJA = A$$
$$AJO = O$$
$$AJ(B+C) = B+C = AJB + AJC$$
$$AJ(AJB) = AJB.$$

Hence J_1 is independent of the other axioms.

EXAMPLE 2. Define AJB as $A \times B \times K$ where K is a given set.

Then
$$AJB = A \times B \times K = BJA$$
$$AJO = A \times O \times K = O$$
$$AJA = A \times A \times K = AK \neq A$$
$$AJ(B+C) = A \times (B+C)K = AJB + AJC$$
$$AJ(AJB) = A.A.B.K.K = AJB.$$

Here J_2 is not satisfied while all the rest are satisfied and therefore J_2 is independent of the rest.

EXAMPLE 3. Define AJB as $A+B$.

Then
$$AJB = A+B = BJA$$
$$AJA = A+A = A$$
$$AJO = A+O = A \neq O$$
$$AJ(B+C) = A+B+C = AJB + AJC$$
$$AJ(AJB) = A+A+B = AJB.$$

Hence $AJO = O$ cannot be deduced from the other axioms.

EXAMPLE 4. Consider a space consisting of three elements (a, b, c) in which the closure operation is defined as follows:

$$\overline{O} = O, \ \overline{a} = a, \ \overline{b} = b, \ \overline{c} = c \ \text{and} \ \overline{X} = 1$$

for other sets X.

This space satisfies only the last three axioms of Kuratowski and if we define AJB as $A\overline{B}+B\overline{A}$ in this space, then by a remark made at the end of **3**, it follows that this space satisfies all the axioms of the set (J) except J_4.

This proves the independence of J_4.

EXAMPLE 5. Here also we take our space to consist of three elements (a,b,c) and the closure operation is defined as follows: $\overline{O} = O$, $\overline{a} = (a, b)$, $\overline{b} = (b, c)$, $\overline{c} = (c, a)$, and $\overline{X} = 1$ for other sets X. This space satisfies only K_1, K_2 and K_4 and as in the previous example, if we take AJB as $A\overline{B}+B\overline{A}$, then it follows by the same remark made at the end of **3** that all the axioms of the set (3) except J_5 are true in this space. Therefore J_5 is independent of the other axioms.

A NOTE ON CYCLIDES

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1. In this paper, some properties of the general cyclide are investigated, regarding the existence of a centre, and also regarding some loci associated with the polar quadrics of the surface.

A cyclide is a quartic surface, having the circle at infinity as a double curve. It can also be generated as the envelope of a two-parameter family of spheres whose centres move on a fixed quadric, called a *focal quadric*, and which intersect orthogonally a fixed sphere, called a *sphere of inversion*. From this mode of generation, the Cartesian equation of a cyclide may be written in the form

$$F(x, y, z) \equiv (r^2+\mu^2)^2-4[a^2(x-\alpha)^2+b^2(y-\beta)^2+c^2(z-\gamma)^2]=0, \tag{1}$$

where $\quad r^2 = x^2+y^2+z^2$ and $\mu^2 = k^2-\alpha^2-\beta^2-\gamma^2$,

a focal quadric S and the associated sphere of inversion T being taken as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ and } (x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=k^2.$$

This mode of generation can in the general case be varied in four other ways by taking any one of four other focal quadrics S_1, S_2, S_3 and S_4 and the corresponding sphere of inversion T_1, T_2, T_3 and T_4 where*

$$S_r \equiv \frac{x^2}{a^2+\lambda_r}+\frac{y^2}{b^2+\lambda_r}+\frac{z^2}{c^2+\lambda_r}-1=0,$$

$$T_r \equiv (x-\alpha_r)^2+(y-\beta_r)^2+(z-\gamma_r)^2-k_r^2=0, \ (r=1, 2, 3, 4),$$

the several constants being determined by the following equations:

$$\frac{\alpha^2}{a^2+\lambda}+\frac{\beta^2}{b^2+\lambda}+\frac{\gamma^2}{c^2+\lambda}=1+\frac{k^2}{\lambda}$$

$$\alpha_r = \frac{a^2\alpha}{a^2+\lambda_r}, \ \beta_r = \frac{b^2\beta}{b^2+\lambda_r}, \ \gamma_r = \frac{c^2\gamma}{c^2+\lambda_r}$$

* Salmon (Rogers), *Analytical Geometry of Three Dimensions*, Vol. 2, Art. 561 (a).

and $k_r^2 = k^2 + \lambda_r \left[2 - \frac{(2a^2+\lambda_r)\alpha^2}{(a^2+\lambda_r)^2} - \frac{(2b^2+\lambda_r)\beta^2}{(b^2+\lambda_r)^2} - \frac{(2c^2+\lambda_r)\gamma^2}{(c^2+\lambda_r)^2} \right].$

2. An algebraic surface of order n will have a centre C, provided that its Cartesian equation referred to C as origin admits of the form

$$u_n + u_{n-2} + u_{n-4} + \ldots = 0,$$

where u_m is homogeneous of degree m in x, y, z. The conditions that the cyclide given by (1) should have a centre are easily seen to be $\alpha = \beta = \gamma = 0$. When these conditions are satisfied, the focal quadrics S_1, S_2, S_3 and S_4 all coincide with S, while the spheres T_1, T_2, T_3 and T_4 coincide with T. Thus

A cyclide has in general no centre. When however a cyclide possesses a centre, say C, it will have only one focal quadric and only one sphere of inversion, having C for their common centre. Conversely if a focal quadric is concentric with its corresponding sphere of inversion, the common centre will also be the centre of the cyclide, and at the same time the five focal quadrics will coincide and the corresponding spheres of inversion will also coincide.

The standard form of a central cyclide may be written as

$$(x^2+y^2+z^2+k^2)^2 = A^2x^2+B^2y^2+C^2z^2.$$

3. There are some interesting loci obtained from considerations of the polar quadrics of a cyclide. We first write down the following results:

Let $F(x, y, z) = 0$ be the equation of any surface referred to rectangular Cartesian co-ordinates, and let the equation be written in the homogeneous form

$$\phi(x, y, z, w) = 0,$$

where $$\phi(x, y, z, 1) \equiv F(x, y, z).$$

The polar quadric of any point (x_1, y_1, z_1, w_1) is given by

$$\left(x\frac{\partial}{\partial x_1} + y\frac{\partial}{\partial y_1} + z\frac{\partial}{\partial z_1} + w\frac{\partial}{\partial w_1} \right)^2 \phi(x_1, y_1, z_1, w_1) = 0.$$

(1) The locus of a point whose polar quadric is a paraboloid is in general a surface of order $3(n-2)$, given by

$$\begin{vmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{vmatrix} = 0.$$

(2) The locus of a point whose polar quadric is *equilateral,** i.e. whose asymptotic cone possesses triads of mutually perpendicular generators, is the surface

$$F_{xx}+F_{yy}+F_{zz}=0.$$

(3) The locus of a point whose polar quadric is *orthogonal**, i.e. whose asymptotic cone possesses triads of mutually perpendicular tangent planes, is the surface

$$F_{yy}F_{zz}+F_{zz}F_{xx}+F_{xx}F_{yy}-F_{yz}^2-F_{zx}^2-F_{xy}^2=0.$$

(4) The locus of a point whose polar quadric is a surface of revolution is in general a curve defined by

$$F_{xx}-\frac{F_{zx}F_{xy}}{F_{yz}}=F_{yy}-\frac{F_{xy}F_{yz}}{F_{zx}}=F_{zz}-\frac{F_{yz}F_{zx}}{F_{xy}},$$

together with the three sets of isolated points determined by

$$F_{zx}=F_{xy}=0,\ (F_{xx}-F_{yy})(F_{xx}-F_{zz})=F_{yz}^2$$

and two similiar triads of equations.

It is possible that all the results (1)—(4) become modified when the surface is of a specialized nature.

(5) In general, no polar quadric will be a sphere. Let us examine whether there is any (special) surface, all of whose polar quadrics are spheres. The conditions

$$F_{xx}=F_{yy}=F_{zz} \qquad (a)$$

$$F_{yz}=F_{zx}=F_{xy}=0 \qquad (b)$$

should be identically satisfied. The differential equations (b) lead to

$$F=f_1(x,y)+f_2(x,z)=f_3(y,z)+f_4(y,x)=f_5(z,x)+f_6(z,y),$$

so that

$$F=\theta(x)+\phi(y)+\psi(z),$$

where θ, ϕ, ψ are arbitrary functions. Then (a) gives

$$\theta''(x)=\phi''(y)=\psi''(z).$$

These equations can be true for all values of x, y, z only if

$$\theta''(x)=\phi''(y)=\psi''(z)=\text{a constant } 2\lambda, \text{ say}.$$

* The term 'equilateral' was given by Schröter: *Oberflachen Zweiter Ordnung* (1880). See also Salmon: *Analytical Geometry of Three Dimensions* (1912), Vol. 1, p. 116. Ex. 8. Sommerville uses 'rectangular' for 'equilateral', and gives the above meaning for an *orthogonal* quadric. Vide his *Analytical Geometry of Three Dimensions*, pp. 102-3, also 322-3.

We then get

$$F = \lambda(x^2+y^2+z^2)+2Ax+2By+2Cz+D,$$

i. e. the surface is a sphere. Omitting this trivial solution, we can therefore say that

There is no algebraic surface for which every polar quadric is a sphere.

4. We shall now apply these results to the cyclide given by equation (1).

(*a*) The determinant in § 3(1) simplifies for the cyclide to the form

$$r^6+u_0r^4+u_2r^2+v_2+v_0,$$

where $r^2=x^2+y^2+z^2$, and u_0, v_0, u_2, v_2 are homogeneous functions of x, y, z of degrees given by the suffixes. Hence,

The locus of a point whose polar quadric with respect to a cyclide is a paraboloid is a CENTRAL *sextic surface, having the circle at infinity as a triple curve.*

(*b*) *The locus of a point whose polar quadric with respect to a cyclide is equilateral is a sphere.*

(*c*) The equation in § 3(3) simplifies for the cyclide to

$$3(r^2+\mu^2)^2+4(r^2+\mu^2)(r^2-a^2-b^2-c^2)$$
$$+4\{b^2c^2+c^2a^2+a^2b^2-(b^2+c^2)x^2-(c^2+a^2)y^2-(a^2+b^2)z^2\}=0.$$

This represents a cyclide which possesses a centre. Hence,

The locus of a point whose polar quadric with respect to a cyclide is orthogonal is a CENTRAL *cyclide.*

(*d*) When $a \neq b \neq c$, it is easily verified that

The locus of a point whose polar quadric with respect to a cyclide is a quadric of revolution consists of the three focal conics of the focal quadric associated with the cyclide.

When, however, $a=b=c$, the cyclide assumes the special form

$$(r^2+\mu^2)^2-4a^2[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]=0.$$

This surface possesses the property that *every polar quadric is a quadric of revolution.*

5. We may observe that the characteristic property in the generation of a cyclide is that the locus of the centres of the enveloping spheres is a quadric, and that the spheres all intersect

orthogonally a fixed sphere. The existence of ∞^2 of enveloping spheres is however possible for every surface. In other words,

Any algebraic surface can in general be regarded as the envelope (partial or complete) of a finite number of families of two-parameter systems of spheres.

This will be evident from the fact that the general equation of a sphere involves four constants, and two conditions are required in order that it may touch a given algebraic surface at two distinct points.

We may write down here the differential equations of the five systems of ∞^2 of spheres which have double contact with the cyclide. The spheres of one system are given by

$$2a(x-\alpha)\sin u\cos v+2b(y-\beta)\sin u\sin v+2c(z-\gamma)\cos u = x^2+y^2+z^2+\mu^2, \quad (3)$$

where u, v are two independent parameters and $\mu^2=k^2-\alpha^2-\beta^2-\gamma^2$. Eliminating u and v, we arrive at the differential equation

$$\psi(x, y, z, p, q)=0, \quad (4)$$

where

$$\psi \equiv b^2\{(x+pz)V-cpU\}^2+a^2\{(y+qz)V-cqU\}^2 - a^2b^2(V^2-U^2),$$

$$U\equiv 2(x-\alpha)(x+pz)+2(y-\beta)(y+qz)-(x^2+y^2+z^2+\mu^2),$$

$$V\equiv 2cp(x-\alpha)+2cq(y-\beta)-2c(z-\gamma).$$

Equation (3) represents a "complete integral" of the differential equation (4), whose singular solution is (1) itself. If we form the "general integral" of (4), by the usual method, any surface so obtained will possess the property of having one set of lines of curvature as circles*.

The differential equations corresponding to the other four systems of spheres associated with a non-central cyclide are obtained by replacing the constants $(a, b, c, \alpha, \beta, \gamma, k)$ by

$$(a^2+\lambda_r)^{\frac{1}{2}},\ (b^2+\lambda_r)^{\frac{1}{2}},\ (c^2+\lambda_r)^{\frac{1}{2}},\ \alpha_r,\ \beta_r,\ \gamma_r,\ k_r,\ (r=1, 2, 3, 4).$$

* Vide Goursat-Hedrick: *Mathematical Analysis*, Vol. I, Art. 252.

A CORRECTION

By K. SAMBASIVA RAO.

In my paper* 'On a particular representation of integers as sums of kth powers', the validity of the proof of Theorem 1 has been questioned in the review appearing in *Mathematical Reviews*. The proof can be made rigorous by making the following corrections.

(1) Insert the following after line 24, p. 263:

Now we shall argue with the least of such r, viz.,

$$\left[\frac{\log\log x - \log\log(1+1/2^{k^2})}{\log\ (k/k-1)}\right]+1.$$

(2) Omit line 3, p. 264.

* *Jour. Indian Math. Soc.*, (2) 3 (1939), 262-5.

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